Alternative Formulations for Discrete Gyroscopic Eigenvalue Problems

Raymond H. Plaut*
Virginia Polytechnic Institute and State University, Blacksburg, Va.

Eigenvalue problems associated with discrete gyroscopic systems have the property that the characteristic equation is a function of the square of the eigenvalue. It may be advantageous, therefore, to replace the given system by an equivalent one that only involves this squared parameter. Meirovitch has recently derived such an equivalent problem for a certain class of gyroscopic systems. The original problem, involving symmetric and skew-symmetric matrices, can be replaced by one involving only symmetric matrices. These results are extended to a more general class of systems in the present paper. In addition, a particular form of gyroscopic system is considered that occurs in problems such as rotating shafts and fluid-conveying pipes. For these systems an equivalent formulation is obtained which leads to a new condition for flutter instability and a technique for determining approximations to frequencies of vibration. An example is presented to illustrate these features.

Introduction

In a recent paper, Meirovitch[1] considered discrete gyroscopic systems that are governed by eigenvalue problems of the form

$$(\lambda J+G)x=0 \tag{1}$$

where J is a $2n\times 2n$ real symmetric nonsingular matrix, G is a $2n\times 2n$ real skew-symmetric matrix, x is a $2n$-dimensional vector, and 0 is the $2n$-dimensional null vector. The characteristic equation

$$|\lambda J+G|=0 \tag{2}$$

is a function of λ^2, and therefore it should be possible to find an equivalent eigenvalue problem that only involves the parameter λ^2. Meirovitch accomplished this for the case when all eigenvalues λ are pure imaginary, obtaining a characteristic equation of the form

$$|\lambda^2 J+K|=0 \tag{3}$$

where J and K are both real and symmetric. Each eigenvalue λ of Eq. (2) appears twice as an eigenvalue λ of Eq. (3). Then, assuming that J is positive definite, he showed that λ^2 could be determined as the eigenvalues of a single real symmetric matrix, thus transforming Eq. (2) into a standard form to which many well-known algorithms can be applied.

In Ref. 1 Meirovitch also considered a special type of gyroscopic system which is associated with the eigenvalue problem

$$(\lambda^2 m+\lambda g+k)r=0 \tag{4}$$

where m is an $n\times n$ real symmetric nonsingular matrix, g is an $n\times n$ real skew-symmetric nonsingular matrix, and k is an $n\times n$ real symmetric matrix. Again the characteristic equation

$$|\lambda^2 m+\lambda g+k|=0 \tag{5}$$

is a function of λ^2 and can be transformed into the form of Eq. (2). When m and k are positive definite, J is also positive definite and the eigenvalue problem, Eq. (5), can be transformed into the standard eigenvalue problem involving a single real symmetric matrix.

Meirovitch was motivated in this work by problems of spinning bodies containing elastic parts. The free vibrations of such systems can be related to equations of the form of Eq. (1) or (4). In Ref. 2, Meirovitch applied these techniques to determine the response of rotating structures to external excitation. In both these papers he was interested in systems operating below the lowest critical speed, in which case the matrix J in Eq. (1) and the matrices m and k in Eq. (4) are positive definite. It is of interest to obtain similar types of results for gyroscopic systems operating in a range of stability above the initial critical value (such as a shaft rotating above the first critical speed), or under conditions at which divergence instability (exponentially increasing motion) or flutter instability (vibrations with increasing amplitude) can occur.

In Theorem 1 it is shown that the eigenvalue problem (2) can be transformed to the form of Eq. (3), even if the eigenvalues λ are not all pure imaginary, and that the roots λ^2 are the eigenvalues of the matrix $(J^{-1}G)^2$. Similar results for systems of the type (4) are given in Theorem 2. The special case of $g=\xi h$ and $k=(u-\xi^2 e)$ is then considered, where ξ is a scalar parameter. Rotating shafts and fluid-conveying pipes are examples of systems involving this type of parameter, with ξ representing a speed of rotation or flow.[3] The characteristic equation

$$|\lambda^2 m+\lambda\xi h+u-\xi^2 e|=0 \tag{6}$$

is a function of λ^2 and ξ^2, and Theorem 3 demonstrates how an equivalent eigenvalue problem in terms of these parameters can be obtained. Based on this result, a condition for the onset of flutter instability is derived and an approximate technique for determining vibration frequencies is proposed.

Analysis

The eigenvalue problem (1) is obtained from dynamical systems governed by the set of linear differential equations

$$J\dot{p}(t)+Gp(t)=0 \tag{7}$$

when the motion is assumed to have the form

$$p(t)=x\exp(\lambda t) \tag{8}$$

Received March 12, 1975; revision received September 29, 1975. This research was supported by the National Science Foundation under Grant GK-41223 and the Army Research Office, Durham under Grant DA-ARO-D-31-124-73-G130 to Brown University. The author wishes to thank the reviewers for their helpful comments.
Index category: Structural Dynamic Analysis.
*Professor, Department of Civil Engineering.

Here J and G are real $2n \times 2n$ matrices while p and x are $2n$-dimensional vectors. Attention is focused on gyroscopic systems for which J is symmetric and nonsingular while G is skew-symmetric. The eigenvalues can be computed from the characteristic Eq. (2). They will be pure imaginary if J is positive definite, but in general they may be of real or complex.† Since the characteristic equation is a function of λ^2, it is possible to derive some equivalent eigenvalue problems in terms of this parameter. Each of the $2n$ eigenvalues λ of Eq. (2) appears twice among the $4n$ eigenvalues λ of the alternative problems. These results are presented in the following theorem:

Theorem 1‡: Consider the system

$$(\lambda J + G)x = 0 \tag{9}$$

where J is real, symmetric and nonsingular while G is real and skew-symmetric.

a) The eigenvalues λ of Eq. (9) are also the eigenvalues λ of the symmetric system

$$(\lambda^2 J + K)y = 0 \tag{10}$$

where

$$K = -GJ^{-1}G \tag{11}$$

with each eigenvalue of Eq. (9) appearing twice as an eigenvalue of Eq. (10).

b) The eigenvalues λ of Eq. (9) are the square roots of the eigenvalues of the matrix L, i.e.,

$$|L - \lambda^2 I| = 0 \tag{12}$$

where

$$L = (J^{-1}G)^2 \tag{13}$$

and I denotes the identity matrix.

c) If J is also positive definite, then the eigenvalues λ of Eq. (9) are the square roots of the eigenvalues of the symmetric matrix V, where

$$V = J^{-1/2} G J^{-1} G J^{-1/2} \tag{14}$$

Meirovitch[1] proved part a when all eigenvalues are pure imaginary and also part b. To prove part a in general, note that

$$|\lambda^2 J - GJ^{-1}G| = |(\lambda J - G)J^{-1}(\lambda J + G)|$$

$$= |\lambda J - G| \cdot |J^{-1}| \cdot |\lambda J + G| \tag{15}$$

and therefore, since[1]

$$|\lambda J - G| = |\lambda J + G| \tag{16}$$

it follows that

$$|\lambda^2 J - GJ^{-1}G| = 0 \tag{17}$$

implies Eq. (2). Parts b and c follow immediately, since

$$|\lambda^2 J - GJ^{-1}G| = |J| \cdot |\lambda^2 I - J^{-1}GJ^{-1}G| \tag{18}$$

and, if J is positive definite,

$$|\lambda^2 J - GJ^{-1}G| =$$

$$|J^{1/2}| \cdot |\lambda^2 I - J^{-1/2}GJ^{-1}GJ^{-1/2}| \cdot |J^{1/2}| \tag{19}$$

†Even though these systems are conservative, flutter instability is possible due to the presence of gyroscopic terms.[3]

‡This theorem is also valid when system (9) is of odd order, in which case the characteristic equation (2) has the form $\lambda f(\lambda^2) = 0$.

The alternative eigenvalue problems given in Theorem 1 may have computational advantages over the direct use of Eq. (2). In Eq. (10), for example, the matrices J and K are symmetric, and V is symmetric in Eq. (14). If the system is in a stable regime, such as when J is positive definite, then the roots λ of Eq. (2) are pure imaginary but the roots λ^2 of the alternative problems are real. In fact, the roots λ^2 will be real except when flutter instability can occur in the system. Since the characteristic Equation (2) is a function of λ^2, it appears that λ^2 should be the appropriate parameter to use in these gyroscopic problems, and Theorem 1 provides the framework for doing this.§

Certain gyroscopic systems are governed by equations of motion of the form

$$m\ddot{q}(t) + g\dot{q}(t) + kq(t) = 0 \tag{20}$$

where m, g, and k are real $n \times n$ matrices, with m and k symmetric and g skew-symmetric, and q is an n-dimensional vector. When

$$q(t) = r \exp(\lambda t) \tag{21}$$

one obtains the eigenvalue problem (4) and the characteristic equation (5). By introducing the state vector

$$x = \begin{bmatrix} \lambda r \\ r \end{bmatrix} \tag{22}$$

one can transform Eq. (4) to the form (1), in which J and G are $2n \times 2n$ matrices that can be defined by

$$J = \left[\begin{array}{c|c} m & 0 \\ \hline 0 & b \end{array}\right], \qquad G = \left[\begin{array}{c|c} g & k \\ \hline -b & 0 \end{array}\right] \tag{23}$$

where b is any nonsingular $n \times n$ matrix. Here J and G are not necessarily symmetric and skew-symmetric, respectively, but the characteristic equation (2) is still a function of λ^2 due to the form of Eq. (4). Alternative problems in terms of λ^2 can be obtained again, and they are stated in the following theorem:

Theorem 2: Consider the system

$$(\lambda^2 m + \lambda g + k)r = 0 \tag{24}$$

where m is real, symmetric and nonsingular, g is real, skew-symmetric and nonsingular, and k is real and symmetric.

a) The eigenvalues λ of Eq. (24) are also the eigenvalues λ of the system

$$(\lambda^2 J + K)y = 0 \tag{25}$$

where

$$J = \left[\begin{array}{c|c} m & 0 \\ \hline 0 & b \end{array}\right], \qquad K = \left[\begin{array}{c|c} k - gm^{-1}g & -gm^{-1}k \\ \hline bm^{-1}g & bm^{-1}k \end{array}\right] \tag{26}$$

and b is any nonsingular matrix, with each eigenvalue of Eq. (24) appearing twice as an eigenvalue of Eq. (25).

b) The eigenvalues λ of Eq. (24) are the square roots of the eigenvalues of the matrix L, where

$$L = \left[\begin{array}{c|c} m^{-1}(gm^{-1}g - k) & m^{-1}gm^{-1}k \\ \hline -m^{-1}g & -m^{-1}k \end{array}\right] \tag{27}$$

§The *eigenvectors* of the original system can be determined easily if the eigenvalues and eigenvectors of the alternative problem have been computed (see the Appendix).

c) If m is also positive definite, then the eigenvalues λ of Eq. (24) are the square roots of the eigenvalues of the matrix V, where

$$V=\begin{bmatrix} m^{-1/2}(gm^{-1}g-k)m^{-1/2} & m^{-1/2}gm^{-1}kc^{-1} \\ -cm^{-1}gm^{-1/2} & -cm^{-1}kc^{-1} \end{bmatrix} \tag{28}$$

and c is any nonsingular matrix.

To prove part a, let

$$y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{29}$$

where y_1 and y_2 are n-dimensional vectors. Then Eq. (25) yields the equations

$$(\lambda^2 m+k-gm^{-1}g)y_1-gm^{-1}ky_2=0 \tag{30a}$$

$$bm^{-1}g\, y_1+(\lambda^2 b+bm^{-1}k)y_2=0 \tag{30b}$$

From Eq. (30b),

$$y_1=-g^{-1}(\lambda^2 m+k)y_2 \tag{31}$$

and substitution of this expression into Eq. (30a) leads to the equation

$$-(\lambda^2 m-\lambda g+k)g^{-1}(\lambda^2 m+\lambda g+k)y_2=0 \tag{32}$$

Since

$$|\lambda^2 m-\lambda g+k| = |(\lambda^2 m-\lambda g+k)^T| = |\lambda^2 m+\lambda g+k| \tag{33}$$

it follows from Eq. (32) that the (repeated) eigenvalues λ of Eq. (25) are the same as those of Eq. (24). The relationship

$$|\lambda^2 I-L| = |\lambda^2 I+J^{-1}K| = |J^{-1}| \cdot |\lambda^2 J+K| \tag{34}$$

leads to part b. If one sets $b=c^2$ in Eq. (26), one can show that

$$V=-J^{-1/2}KJ^{-1/2} \tag{35}$$

and therefore

$$|\lambda^2 I-V| = |J^{-1/2}| \cdot |\lambda^2 J+K| \cdot |J^{-1/2}| \tag{36}$$

which proves c.

In Theorem 2 the nonsingular $n\times n$ matrices b and c may be chosen arbitrarily. For example, using m for b yields

$$J=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad K=\begin{bmatrix} k-gm^{-1}g & -gm^{-1}k \\ g & k \end{bmatrix} \tag{37}$$

If b and c are selected as the identity matrix, then they simply can be deleted from the expressions for K and V in Theorem 2. If k is nonsingular, the choice $b=k$ makes J and K symmetric, while V can be made symmetric if k is positive definite by choosing $c=k^{1/2}$; these particular values were used in Ref. 1.

For some gyroscopic systems the equations of motion (20) have the special form

$$m\ddot{q}(t)+\xi h\dot{q}(t)+uq(t)-\xi^2 eq(t)=0 \tag{38}$$

where m, h, u, and e are real $n\times n$ matrices, with m, u, and e symmetric and h skew-symmetric, and ξ is a scalar parameter. Equation (6) gives the associated characteristic equation when motion of the form (21) is assumed. Since this characteristic equation is a function of λ^2 and ξ^2, it is natural to seek alternative problems involving these parameters. Two such formulations are given in the following theorem:

Theorem 3 Consider the system

$$(\lambda^2 m+\lambda\xi h+u-\xi^2 e)r=0 \tag{39}$$

where m is real, symmetric, and nonsingular, h is real, skew-symmetric, and nonsingular, and u and e are real and symmetric.

a) The eigenvalues λ of Eq. (39) are also the eigenvalues λ of the system

$$(\lambda^2 A+\xi^2 B+C)y=0 \tag{40}$$

where

$$A=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad B=\begin{bmatrix} -(e+hm^{-1}h) & hm^{-1}e \\ h & -e \end{bmatrix}$$

$$C=\begin{bmatrix} u & -hm^{-1}u \\ 0 & u \end{bmatrix} \tag{41}$$

with each eigenvalue of Eq. (39) appearing twice as an eigenvalue of Eq. (40).

b) If m is also positive definite, then the eigenvalues λ of Eq. (39) are the square roots of the eigenvalues of the matrix W, where

$$W=D+\xi^2 E \tag{42}$$

$$D=\begin{bmatrix} -m^{-1/2}um^{-1/2} & m^{-1/2}hm^{-1}ua^{-1} \\ 0 & -am^{-1}ua^{-1} \end{bmatrix}$$

$$E=\begin{bmatrix} m^{-1/2}(e+hm^{-1}h)m^{-1/2} & -m^{-1/2}hm^{-1}ea^{-1} \\ -am^{-1}hm^{-1/2} & am^{-1}ea^{-1} \end{bmatrix} \tag{43}$$

and a is any nonsingular matrix.

To prove Theorem 3, first assume $\xi\neq 0$ and let y have the form (29). Equation (40) leads to the relations

$$y_1=-\frac{1}{\xi^2}h^{-1}(\lambda^2 m+u-\xi^2 e)y_2 \tag{44}$$

and

$$-\frac{1}{\xi^2}(\lambda^2 m-\lambda\xi h+u-\xi^2 e)h^{-1}(\lambda^2 m +\lambda\xi h+u-\xi^2 e)y_2=0 \tag{45}$$

Hence the eigenvalues of Eq. (40) satisfy Eq. (6). Part b follows directly from Theorem 2 c with $g=\xi h$, $k=u-\xi^2 e$, and the choice $c=\xi a$. If $\xi=0$, Eq. (40) yields

$$(\lambda^2 m+u)y_1-hm^{-1}uy_2=0 \tag{46a}$$

$$(\lambda^2 m+u)y_2=0 \tag{46b}$$

and therefore the eigenvalues of system (40) satisfy

$$|\lambda^2 m+u|=0 \tag{47}$$

which is the same as Eq. (6). A similar proof shows that part b is also valid when $\xi=0$.

For gyroscopic systems of the form (38), the total energy $\mathcal{E}$ is given by

$$\mathcal{E}=\tfrac{1}{2}(\dot{q}^T m\dot{q}+q^T uq-\xi^2 q^T eq) \tag{48}$$

During motion,

$$d\mathcal{E}/dt = 0 \tag{49}$$

and therefore the system is conservative. Due to the gyroscopic matrix h, however, flutter instability may occur for certain values of ξ, and it is of interest to determine a "flutter condition" that is satisfied at the onset of flutter. Consider the form (40) and let $\lambda^2 = -\omega^2$, so that

$$(-\omega^2 A + \xi^2 B + C)y = 0 \tag{50}$$

Also, let z denote the left eigenvector, i.e.,

$$z^T(-\omega^2 A + \xi^2 B + C) = 0 \tag{51}$$

Assuming $z^T By \neq 0$, the expression

$$\xi^2 = \frac{\omega^2 z^T Ay - z^T Cy}{z^T By} \tag{52}$$

is stationary with respect to variations in y and z if Eqs. (50) and (51) are satisfied.[4] Now consider the "loading-frequency" characteristic curves in the (ω^2, ξ^2) plane.[3] At the onset of flutter instability, two vibration frequencies coalesce and the corresponding characteristic curve has a zero slope. Hence, for the appropriate eigenvectors y and z, the slope expression[5]

$$\frac{d(\xi^2)}{d(\omega^2)} = \frac{z^T Ay}{z^T By} \tag{53}$$

leads to the flutter condition

$$z^T Ay = 0 \tag{54}$$

One can also make use of the slope expression (53) to determine approximate values of vibration frequencies. This will be illustrated by an example.

Example

Consider a circular, elastic, simply supported pipe of mass per unit length μ conveying a fluid of mass per unit length δ at constant velocity v. Let x denote the axial coordinate, t the time, $w(x,t)$ the transverse displacement, s the bending stiffness, and ℓ the length of the pipe. The equation of motion is assumed to be[6]

$$s\frac{\partial^4 w}{\partial x^4} + v^2\delta\frac{\partial^2 w}{\partial x^2} + 2v\delta\frac{\partial^2 w}{\partial x \partial t} + (\mu+\delta)\frac{\partial^2 w}{\partial t^2} = 0 \tag{55}$$

Define the nondimensional quantities

$$\xi^2 = \frac{v^2\ell^2\delta}{\pi^2 s}, \quad \omega^2 = \frac{\Omega^2\ell^4(\mu+\delta)}{\pi^4 s}, \quad \gamma = \frac{\delta}{\mu+\delta} \tag{56}$$

where Ω is the frequency of vibration. Using the Galerkin procedure with the first two modes of free vibration yields a set of equations of the form (39) with $\lambda = i\omega$,

$$m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$$

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \tag{57}$$

where $\alpha^2 = 256\gamma/(9\pi^2)$. Characteristic curves in the (ω^2, ξ^2) plane for the cases $\alpha^2 = 0.5$, 1.0, and 1.5 can be found in Ref. 3.

For the alternative system (50), Eq. (41) gives $A = I$,

$$B = \begin{bmatrix} (\alpha^2-1) & 0 & 0 & -4\alpha \\ 0 & (\alpha^2-4) & \alpha & 0 \\ 0 & -\alpha & -1 & 0 \\ \alpha & 0 & 0 & -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 16\alpha \\ 0 & 16 & -\alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} \tag{58}$$

The slope expression (53) can be used to determine approximate values of ω^2. For example, consider the fundamental characteristic curve in the first quadrant of the (ω^2, ξ^2) plane. An approximation to this curve can be obtained by drawing the tangents to the curve where it intercepts the ω^2 and ξ^2 axes. From Eq. (6), the intercept at $\xi^2 = 0$ is found to be $\omega^2 = 1$. Equations (50) and (51) yield the corresponding eigenvectors y and z, respectively, and Eq. (53) then gives the value of the slope as

$$\frac{d(\xi^2)}{d(\omega^2)} = -\frac{15}{(15+\alpha^2)} \tag{59}$$

Similarly, the ξ^2 intercept is $\xi^2 = 1$ and the slope there is found to be

$$\frac{d(\xi^2)}{d(\omega^2)} = -\frac{(12+\alpha^2)}{12} \tag{60}$$

Connecting these two tangents yields a piecewise linear approximation to the fundamental characteristic curve, as shown in Fig. 1 for the case $\alpha^2 = 1$. (The dashed line segment connecting the intercepts in Fig. 1 corresponds to the case $h = 0$, i.e., $\alpha = 0$, and hence is an upper bound to this curve).[3] Based on Theorem 3, therefore, one can obtain an approximation to the fundamental vibration frequency for all fluid velocities below the critical value, simply by computing and utilizing the appropriate eigenvalues and eigenvectors for the case $\xi^2 = 0$ and the case $\omega^2 = 0$.

Conclusions

It is shown that eigenvalue problems for discrete gyroscopic systems can be transformed to equivalent problems involving

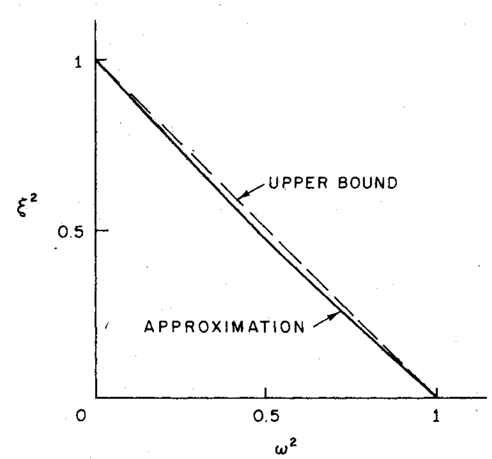

Fig. 1 Fundamental characteristic curve for fluid-conveying pipe.

the square of the eigenvalue. These alternative problems may be more efficient for computational purposes. In some cases they only contain symmetric matrices. Often they involve real parameters while the original problem has complex solutions. For the gyroscopic systems considered in Theorem 2, there is a matrix in the alternative formulation which is free to be chosen so as to increase the efficiency of a particular algorithm. Finally, for the special class of gyroscopic systems treated in Theorem 3, the equivalent problems involve the squares of both parameters λ and ξ, leading to a convenient representation in the (ω^2,ξ^2) plane, where $\lambda^2=-\omega^2$, and to useful properties of the characteristic curves in that plane.

Appendix

The eigenvectors of the original system can be obtained in the usual manner, substituting the computed eigenvalues into the original equations and solving the resulting set of linear algebraic equations. However, if the eigenvectors of the alternative problem have been found, they can be utilized to determine the original eigenvectors with the solution of just a single equation.

First consider Theorem 1 with the original system having the form (9). For a particular (double) eigenvalue λ_c^2 of problem (10) there will generally exist two linearly independent eigenvectors $y_c^{(1)}$ and $y_c^{(2)}$. The eigenvector x_c corresponding to λ_c in system (9) will be a linear combination of these, i.e.,

$$x_c=d_1y_c^{(1)}+d_2y_c^{(2)} \tag{A1}$$

and the ratio d_1/d_2 can be determined simply by solving any one of the equations obtained when λ_c and this expression for x_c are substituted into Eq. (9). The same procedure is valid for the alternative problem of Theorem 1b, using the eigenvectors of the matrix L. For part c, if the eigenvectors of V are denoted by $\hat{y}_c^{(j)}, j=1,2$, one can find x_c with the use of[1]

$$y_c^{(j)}=J^{-1/2}\hat{y}_c^{(j)}, \quad j=1,2 \tag{A2}$$

and Eqs. (A1) and (9).

If the original system has the form (24) of Theorem 2, one can apply Eq. (A1) to the eigenvectors of Eq. (25) or of the matrix L and then obtain the eigenvector $r=r_c$ from Eq. (22). The procedure for Theorem 2c is similar to that for part c of Theorem 1. When form (39) is applicable, the eigenvectors $y_c^{(j)}$ from Eq. (40) are the same as those from Eq. (10) that are used in Eq. (A1), whereas in Theorem 3b the eigenvectors $\hat{y}_c^{(j)}$ of W are related to $y_c^{(j)}$ by Eq. (A2).

As an example, consider the system (39) with the matrices defined by Eq. (57). Let $\alpha=2$ and $\xi=1$, and choose $b=m$ in Eq. (26). For alternative problem (40) or, equivalently, Eq. (25), one has $J=A=I$ and

$$K=\xi^2B+C=\begin{bmatrix} 4 & 0 & 0 & 24 \\ 0 & 16 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 12 \end{bmatrix} \tag{A3}$$

The double roots of Eq. (3) are $\lambda^2=0$ and $\lambda^2=-16$. Corresponding to $\lambda_c^2=-16$ are the two eigenvectors

$$y_c^{(1)}=\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad y_c^{(2)}=\begin{bmatrix} 0 \\ -8 \\ 1 \\ 0 \end{bmatrix} \tag{A4}$$

Equation (9), with $\lambda=4i$, x defined by Eq. (A1), and G from Eq. (23) as

$$G=\begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 12 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{A5}$$

then yields 4 equations in d_1 and d_2, any one of which gives $d_1=2id_2$. Hence

$$x_c=\begin{bmatrix} 4i \\ -8 \\ 1 \\ 2i \end{bmatrix} \tag{A6}$$

and, from Eq. (22),

$$r_c=\begin{bmatrix} 1 \\ 2i \end{bmatrix} \tag{A7}$$

In a similar manner, the eigenvalue $\lambda_c^2=0$ leads to the eigenvector

$$r_c=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{A8}$$

of the original system.

References

[1] Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, Oct. 1974, pp. 1337-1342.

[2] Meirovitch, L., "A New Modal Method for the Response of Structures Rotating in Space," *Journal of Applied Mechanics*, Vol. 42, No. 2, June 1975, pp. 446-450.

[3] Huseyin, K. and Plaut, R. H., "Transverse Vibrations and Stability of Systems with Gyroscopic Forces," *Journal of Structural Mechanics*, Vol. 3, No. 2, 1974-1975, pp. 163-177.

[4] Lancaster, P., *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, England, 1966, Sec. 2.8.

[5] Rogers, L. C., "Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 8, May 1970, pp. 943-944.

[6] Paidoussis, M. P. and Issid, N. T., "Dynamic Stability of Pipes Conveying Fluid," *Journal of Sound and Vibration*, Vol. **33**, No. 3, 1974, pp. 267-294.